Fourier Analysis 04-16

Review

Thm. Let $f \in M(\mathbb{R})$. Suppose that $\hat{f} \in M(\mathbb{R})$.

Then

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} d\xi. \quad \text{(Fourier Inversion Formula)}$$

and

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi$$

(Plancherel formula)

Application 1: Time-dependent heat equation on the real line

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & x \in \mathbb{R}, \ t > 0 \\ u(x,0) = f(x). \end{cases}$$

Define

$$\mathcal{H}_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \quad x \in \mathbb{R}, \ t > 0$$

(heat kernel on the real line)

Let $\mathcal{S}(\mathbb{R})$ denote the Schwartz space, i.e. the

Collection of $\mathbb{C}$-valued functions $f \in C^\infty(\mathbb{R})$ such that

$$\sup_{x\in\mathbb{R}} |x^k f^{(\ell)}(x)| < \infty \quad \forall\ k, \ell \geq 0$$

Thm Let $f \in S(\mathbb{R})$. Let

$$u(x,t) = f * \mathcal{H}_t(x).$$

Then

① $u \in C^\infty(\mathbb{R}\times\mathbb{R}_+)$, $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ on $\mathbb{R}\times\mathbb{R}_+$ $= (-\infty,\infty)\times(0,\infty)$

② $u(x,t) \rightrightarrows f(x)$ as $t \to 0$

③ $\displaystyle\int_{\mathbb{R}} |u(x,t) - f(x)|^2 dx \to 0$ as $t \to 0$

Q: Are there another solutions to the heat equation?

An example :

Let $\quad u(x,t) = \frac{x}{t} \mathcal{H}_t(x), \quad t>0, \quad x \in \mathbb{R}.$

Check: $\bullet \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \checkmark$

$\bullet\bullet \quad \lim_{t\to 0} u(x,t) = 0 \quad$ for all $x \in \mathbb{R}$.

(If letting $u \equiv 0$, u is also a solution)

<u>Prop 1</u>: Let $f \in \mathcal{S}(\mathbb{R})$ and $u(x,t) = f * \mathcal{H}_t(x)$.

Then $u(\cdot,t)$ belongs to $\mathcal{S}(\mathbb{R})$ uniformly in t in the following sense: given $T>0$,

$$\sup_{\substack{x\in\mathbb{R} \\ 0<t<T}} \left| x^k \cdot \frac{\partial^\ell u(x,t)}{\partial x^\ell} \right| < \infty, \quad \forall k, \ell \geq 0.$$

Proof. Without loss of generality, we prove the result in the case $k=1$, $\ell=1$.

Recall that

$$\frac{\partial u(x,t)}{\partial x} \xrightarrow{\mathcal{F}} (2\pi i\xi)\cdot \hat{u}(\xi,t)$$

$$(-2\pi i x)\frac{\partial u(x,t)}{\partial x} \xrightarrow{\mathcal{F}} \frac{d\left((2\pi i\xi)\,\hat{u}(\xi,t)\right)}{d\xi}$$

Hence by the Fourier inversion formula,

$$-2\pi i x\,\frac{\partial u(x,t)}{\partial x} = \int_{\mathbb{R}} \frac{d\left(2\pi i\xi\,\hat{u}(\xi,t)\right)}{d\xi}\, e^{2\pi i x}\, d\xi$$

Hence

$$\sup_{\substack{x\in\mathbb{R}\\ 0<t<T}} \left|-2\pi i x\,\frac{\partial u(x,t)}{\partial x}\right| \leq \sup_{\substack{x\in\mathbb{R}\\ 0<t<T}} \int_{\mathbb{R}} \left|\frac{d\left(2\pi i\xi\,\hat{u}(\xi,t)\right)}{d\xi}\right| d\xi$$

$$= \sup_{0<t<T} \int_{\mathbb{R}} \left|\frac{d\left(2\pi i\xi\,\hat{u}(\xi,t)\right)}{d\xi}\right| d\xi$$

$$= \sup_{0<t<T} \int_{\mathbb{R}} \left|\underbrace{\frac{d\left(2\pi i\xi\cdot \hat{f}(\xi)\,e^{-4\pi^2\xi^2 t}\right)}{d\xi}}_{(*)}\right| d\xi$$

Notice that

$$(*) = 2\pi i \left[\left(\xi \hat{f}(\xi)\right)' e^{-4\pi^2\xi^2 t} + \xi \hat{f}(\xi) \cdot e^{-4\pi^2\xi^2 t} \left(-8\pi^2 \xi t\right) \right]$$

Hence

$$|(*)| \leq 2\pi \cdot \left[\left|\left(\xi \hat{f}(\xi)\right)'\right| + \left|\xi \hat{f}(\xi)\right| \cdot \left(8\pi^2 t |\xi|\right) \right]$$

Then

$$\sup_{0<t<T} \int_{\mathbb{R}} |(*)| \, d\xi$$

$$\leq 2\pi \int_{\mathbb{R}} \left|\left(\xi \hat{f}(\xi)\right)'\right| d\xi$$

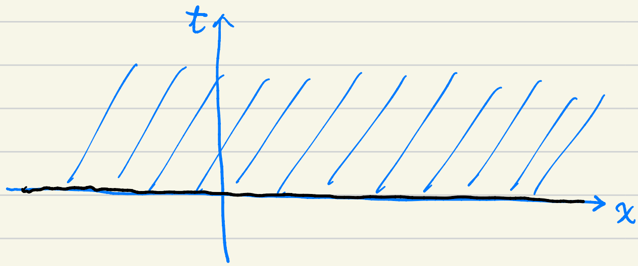
$$+ 16\pi^3 T \int_{\mathbb{R}} |\xi|^2 \left|\hat{f}(\xi)\right| d\xi < \infty.$$

Thm 2 (uniqueness).

Suppose $u = u(x,t)$ satisfies the following conditions:

① $u \in C(\overline{\mathbb{R}\times\mathbb{R}_+}) \cap C^2(\mathbb{R}\times\mathbb{R}_+)$,

where $\overline{\mathbb{R}\times\mathbb{R}_+} = (-\infty,\infty)\times[0,\infty)$.



② $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$ on $\mathbb{R}\times\mathbb{R}_+$

③ $u(x,0) = 0, \quad x\in\mathbb{R}$

④ $u(\cdot, t)$ belongs to $S(\mathbb{R})$ uniformly in t.

Then $u(x,t) \equiv 0$ on $\mathbb{R}\times\mathbb{R}_+$

Proof. (Energy method).

Define for $t \geq 0$,

$$E(t) = \int_{-\infty}^{\infty} |u(x,t)|^2 \, dx.$$

Energy

Since $u(\cdot, t) \in S(\mathbb{R})$ unif in t,

$E(t)$ is finite for any $t \geq 0$.

$E(0) = 0$.

Moreover,

$$\frac{d E(t)}{dt} \overset{(DCT)}{=\!=} \int_{-\infty}^{\infty} \frac{d}{dt} |u(x,t)|^2 \, dx$$

$$= \int_{-\infty}^{\infty} \frac{d}{dt} \left(u(x,t) \cdot \overline{u(x,t)} \right) dx$$

$$= \int_{-\infty}^{\infty} \partial_t u(x,t) \cdot \overline{u(x,t)}$$

$$+ u(x,t) \; \partial_t \overline{u(x,t)} \; dx$$

$$= \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} \cdot \bar{u} + u \cdot \overline{\frac{\partial^2 u}{\partial x^2}} \, dx$$

int by part

$$= -\int_{-\infty}^{\infty} \frac{\partial u}{\partial x} \cdot \frac{\partial \bar{u}}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial \bar{u}}{\partial x} \, dx$$

$$= -2 \int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x} \right|^2 dx$$

$$\leq 0.$$

Hence $E(t)$ is decreasing function in t.

But $E(0) = 0$, so we have

$$E(t) \leq 0 \text{ for } t \geq 0.$$

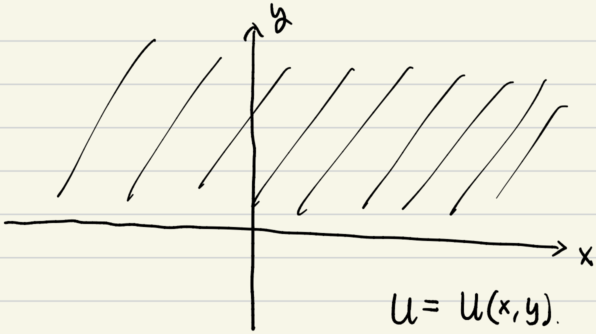
However $E(t) = \int_{-\infty}^{\infty} |u(x,t)|^2 dx$

$\geq 0.$

Hence $E(t) \equiv 0$ for all $t \geq 0$.

It follows that $u(x,t) \equiv 0$. $\blacksquare$

§5.6 Application 2: Steady state heat equation on the upper half plane.



$\mathbb{R} \times \mathbb{R}_+$

$u = u(x,y)$.

$$\begin{cases} \Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 & (1) \\ u(x,0) = f(x). & (2) \end{cases}$$

Takig Fourier transform in x varible in (1),

we obtain

$$(+2\pi i \xi)^2 \cdot \hat{u}(\xi, y) + \frac{\partial^2 \hat{u}(\xi, y)}{\partial y^2} = 0$$

i.e $$\frac{\partial^2 \hat{u}(\xi, y)}{\partial y^2} - 4\pi^2 \xi^2 \hat{u}(\xi, y) = 0$$

The general solution of the above ODE is

$$\hat{u}(\xi, y) = A(\xi) e^{-2\pi|\xi| y} + B(\xi) \cdot e^{2\pi|\xi| y}$$

where $A(\xi)$, $B(\xi)$ are functions in ξ.

Removing the second part (since it is rapidly increasing),

we have

$$\hat{u}(\xi, y) = A(\xi) \cdot e^{-2\pi|\xi| y}.$$

Letting $y=0$,

$$\hat{f}(\xi) = A(\xi)$$

Hence

$$\hat{u}(\xi, y) = \hat{f}(\xi) \cdot e^{-2\pi|\xi|y}.$$

Now let us introduce the Poisson kernel on the upper half plane

$$\mathcal{P}_y(x) = \frac{1}{\pi} \frac{y}{x^2+y^2}, \quad x \in \mathbb{R}, \ y > 0.$$

Claim: $\boxed{\widehat{\mathcal{P}_y}(\xi) = e^{-2\pi|\xi|y}.}$

Then $\hat{u}(\xi, y) = \hat{f}(\xi) \cdot \widehat{\mathcal{P}_y}(\xi)$

$$= \widehat{f * \mathcal{P}_y}(\xi)$$

By Inversion formula, we get

$$u(x,y) = \boxed{f * \mathcal{P}_y(x)}.$$

Lem 3. (i) $\displaystyle\int_{\mathbb{R}} e^{-2\pi|\xi|y} e^{2\pi i \xi x} \, d\xi = P_y(x)$

(ii) $\displaystyle\int_{\mathbb{R}} P_y(x) e^{-2\pi i \xi x} \, dx = e^{-2\pi|\xi|y}$.

Pf. Recall

$$e^{-|x|} \xrightarrow{\mathcal{F}} \frac{2}{1+4\pi^2\xi^2}$$

So

$$e^{-2\pi|x|y} \xrightarrow{\mathcal{F}} \frac{1}{2\pi y} \cdot \frac{2}{1+4\pi^2\left(\frac{\xi}{2\pi y}\right)^2}$$

$$= \frac{1}{2\pi y} \cdot \frac{2}{1+\frac{\xi^2}{y^2}}$$

$$= \frac{1}{\pi} \cdot \frac{y}{\xi^2+y^2}.$$

i.e. $\displaystyle\int_{\mathbb{R}} e^{-2\pi|x|y} e^{-2\pi i \xi x} \, dx$

$$= \frac{1}{\pi} \frac{y}{\xi^2 + y^2}$$

Taking complex conjugate on both sides of the above equation gives

$$\int_{\mathbb{R}} e^{-2\pi|x|y} e^{2\pi i \xi x} \, dx = \frac{1}{\pi} \frac{y}{\xi^2 + y^2}$$

This is the result in (i).

Now (ii) is simply obtained by Inversion formula.

Lem 4: $\{P_y(x)\}_{y>0}$ is a good kernel on $\mathbb{R}$.

Thm 5. Let $f \in S(\mathbb{R})$ and $u(x,y) = f * P_y(x)$.

Then ① $u \in C^2(\mathbb{R} \times \mathbb{R}_+)$ and $\Delta u = 0$

② $u(x,y) \rightrightarrows f(x)$ as $y \to 0$

③ $\int |u(x,y) - f(x)|^2 \, dx \to 0$ as $y \to 0$

④ $u(x,y) \to 0$ as $|x| + y \to \infty$.

Pf. Here we only prove ④. We need to prove

$\exists$ const $C > 0$ such that

$$|u(x,y)| \leq \begin{cases} C\left(\frac{1}{1+x^2} + \frac{y}{x^2+y^2}\right). \\ \frac{C}{y}. \end{cases}$$

$$u = f * P_y(x).$$

$$\text{But } P_y(x) = \frac{1}{\pi} \cdot \frac{y}{x^2+y^2} \leq \frac{1}{\pi} \frac{y}{y^2} = \frac{1}{\pi} \cdot \frac{1}{y}$$

$$f * P_y(x) = \int_{-\infty}^{\infty} f(x-t) P_y(t)\, dt$$

$$= \int_{|t| < \frac{|x|}{2}} + \int_{|t| \geq \frac{|x|}{2}} f(x-t) P_y(t)\, dt$$

$$= ① + ②$$

$$|①| \leq \int_{|t|<\frac{|x|}{2}} |f(x-t)| \, P_y(t)\, dt$$

$$\leq \int_{|t|<\frac{|x|}{2}} \frac{C}{1+\left(\frac{|x|}{2}\right)^2} P_y(t)\, dt$$

$$\leq \frac{4C}{1+|x|^2}$$

$$|②| \leq \int_{|t|\geq\frac{|x|}{2}} |f(x-t)| \, P_y(t)\, dt$$

$$\leq \int_{|t|\geq\frac{|x|}{2}} |f(x-t)| \cdot \frac{1}{\pi} \frac{y}{t^2+y^2}\, dt$$

$$\leq \int_{|t|\geq\frac{|x|}{2}} |f(x-t)| \cdot \frac{1}{\pi} \cdot \frac{4y}{x^2+y^2}\, dt$$

$$\leq \frac{4}{\pi} \frac{y}{x^2+y^2} \int_{\mathbb{R}} |f(t)|\, dt \leq \text{Const.} \frac{y}{x^2+y^2}$$

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